

ON THE SHAPE OF THE GROUND STATE EIGENFUNCTION FOR STABLE PROCESSES

RODRIGO BAÑUELOS, TADEUSZ KULCZYCKI, AND PEDRO J.
MÉNDEZ-HERNÁNDEZ

ABSTRACT. *We prove that the ground state eigenfunction for symmetric stable processes of order $\alpha \in (0, 2)$ killed upon leaving the interval $(-1, 1)$ is concave on $(-\frac{1}{2}, \frac{1}{2})$. We call this property “mid-concavity.” A similar statement holds for rectangles in \mathbb{R}^d , $d > 1$. These result follow from similar results for finite dimensional distributions of Brownian motion and subordination.*

1. INTRODUCTION

Let D be a bounded convex domain in \mathbb{R}^d , $d \geq 1$, and let φ_1 be the first eigenfunction for the Dirichlet Laplacian in D . In their seminal paper [13], Brascamp and Lieb proved that φ_1 is log-concave in D . That is, $\log(\varphi_1)$ is concave on any segment contained in the domain. This result has led to many interesting applications in analysis, geometry, pde, mathematical physics and probability. For some of these applications, see Borell [10], [11], [12] and the many references therein. In particular, the log-concavity of φ_1 leads to estimates of the spectral gap $\lambda_2 - \lambda_1$ which in turn describe the rate to equilibrium of the Brownian motion conditioned to remain forever in the domain D . We refer the reader to [3], [18], [20] and [21] for some of these applications and additional references.

In [4], the first two authors initiated the study of what may be called the “fine spectral theoretic properties” of symmetric stable processes. Unfortunately, given the “nonlocality” of the generator of these processes, even the most basic questions seem to be very difficult. It was proved in [4] (Theorem 5.1) that the ground state eigenfunction for the

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Cauchy process in the interval $(-1, 1)$ is concave. We, of course, expect this to be the case for any symmetric stable process. The purpose of this paper is to prove that for any symmetric stable processes, the ground state eigenfunction is concave in $(-\frac{1}{2}, \frac{1}{2})$. We call this property “*mid-concavity*”. This will follow from a more general result on “mid-concavity” of the finite dimensional distributions of these processes. This “mid-concavity” result is new even for Brownian motion.

We first recall some basic definitions. Let X_t^α be a d -dimensional symmetric stable process of index $0 < \alpha \leq 2$. The process X_t^α has stationary independent increments and its transition density $p_t^\alpha(x, y) = p_t^\alpha(x - y)$, $t > 0$, $x, y \in \mathbb{R}^d$, is determined by its Fourier transform

$$\exp(-t|\xi|^\alpha) = \int_{\mathbb{R}^d} e^{i\xi \cdot y} p_t^\alpha(y) dy.$$

These are Lévy processes with right continuous sample paths. The transition densities satisfy the scaling property

$$p_t^\alpha(x, y) = t^{-d/\alpha} p_1^\alpha(t^{-1/\alpha}x, t^{-1/\alpha}y),$$

hence the process has the scaling property of index α . When $\alpha = 2$, X_t^2 is just Brownian motion B_t running at twice the speed and when $\alpha = 1$, X_t^1 is the Cauchy process. In the first case, $p_t^2(x, y)$ is the usual Gaussian distribution (heat kernel) and in the second, $p_t^1(x, y)$ is the Cauchy distribution (Poisson kernel).

Our interest here is on symmetric stable processes of index $0 < \alpha < 2$ killed upon leaving a domain D . That is, let $D \subset \mathbb{R}^d$, $d \geq 1$, be a nonempty bounded connected open set and let

$$\tau_D^\alpha = \inf\{t \geq 0 : X_t^\alpha \notin D\}$$

be the first exit time of X_t^α from D . Let

$$T_t^D f(x) = E_x(f(X_t^\alpha), \tau_D^\alpha > t),$$

for $x \in D$, $t > 0$ and $f \in L^2(D)$, be the semigroup of the killed process. The killed process has transition densities $p_D^\alpha(t, x, y)$ and

$$(1.1) \quad T_t^D f(x) = \int_D p_D^\alpha(t, x, y) f(y) dy.$$

As with Brownian motion,

$$(1.2) \quad p_D^\alpha(t, x, y) = p^\alpha(t, x, y) - r_D(t, x, y),$$

where

$$(1.3) \quad r_D(t, x, y) = E_x(p_{t-\tau_D^\alpha}^\alpha(X_{\tau_D^\alpha}^\alpha, y), \tau_D^\alpha < t).$$

From this it follows that the transition function $p_D^\alpha(t, x, y)$ is nonnegative, symmetric, jointly continuous in x and y , and that for all $x, y \in D$ and $t > 0$,

$$p_D^\alpha(t, x, y) \leq p_t^\alpha(x, y) = t^{-d/\alpha} p_1^\alpha(t^{-1/\alpha}x, t^{-1/\alpha}y) \leq C t^{-d/\alpha},$$

where $C = (2\pi)^{-d} \omega_d \Gamma(d/\alpha)/\alpha$ and ω_d is the surface measure of the unit sphere in \mathbb{R}^d . In fact, $p_D^\alpha(t, x, y)$ is strictly positive for $x, y \in D$. These properties and the general theory of heat semigroups (as in [15]) gives an orthonormal basis of eigenfunctions $\{\varphi_n^\alpha\}$ on $L^2(D)$ with eigenvalues $\{\lambda_n^\alpha\}$ satisfying $0 < \lambda_1^\alpha < \lambda_2^\alpha \leq \lambda_3^\alpha \leq \dots$, and $\lambda_n^\alpha \rightarrow \infty$, as $n \rightarrow \infty$. That is,

$$T_t^D \varphi_n^\alpha(x) = e^{-\lambda_n^\alpha t} \varphi_n^\alpha(x), \quad x \in D.$$

In addition, the first eigenvalue λ_1^α is simple and its corresponding eigenfunction φ_1^α , which we will refer to as the *ground state eigenfunction*, is an analytic strictly positive function on D . The infinitesimal generator of the semigroup is $-(-\Delta)^{\alpha/2}$. We can think of the eigenfunction and eigenvalues as solutions to the eigenvalue problem

$$(-\Delta)^{\alpha/2} \varphi_n^\alpha(x) = \lambda_n^\alpha \varphi_n^\alpha(x),$$

$x \in D$ and $\varphi_n^\alpha(x) = 0$ for $x \in D^c$; the Dirichlet problem for stable processes. We refer the reader to [5], [7], [9], [14] and [16] where many of the general properties of the α -stable semigroup and its generator are established.

The following question is motivated from the result of Brascamp and Lieb [13] mentioned above for Brownian motion and by its many applications.

Question 1.1. *Let $D \subset \mathbb{R}^d$, $d \geq 1$, be a bounded convex domain and $0 < \alpha < 2$. Is φ_1^α log-concave? In other words, is $\log(\varphi_1^\alpha)$ concave on any segment contained in D ?*

The only known case is when $D = (-1, 1)$ and $\alpha = 1$, where the question is answered in the affirmative in [4]. In fact, it is shown in [4] that the ground state eigenfunction for the Cauchy process in $(-1, 1)$ is concave. Because of this case we believe this result should hold for all α -stable processes. More precisely, we have

Conjecture 1.1. *Let φ_1^α be the ground state eigenfunction for the symmetric stable processes of index $0 < \alpha < 2$ killed upon leaving the interval $I = (-1, 1)$. Then φ_1^α is concave on I .*

There are by now many proofs of the log-concavity result for Brownian motion. None of them, as far as we can see, adapt to the case of general symmetric stable processes. However, Brascamp–Lieb’s proof

does suggest some related questions which may provide some insight. We briefly recall here their argument based on multiple integrals. Let B_t be Brownian motion and let τ_D be its first exit time from D . Then one can show, see [1], that $\varphi_1^2(x) = \lim_{t \rightarrow \infty} e^{\lambda_1^2 t} P_x\{\tau_D > 2t\}$, uniformly in $x \in D$. From this it is enough to prove that $P_x\{\tau_D > t\}$ is log-concave in x for every fixed $t > 0$. The latter can be written as the limit as n and k tend to infinity of $P_x\{B_{jt/n} \in D_k; j = 1, 2, \dots, n\}$ where D_k is a sequence of convex domains strictly increasing ($\overline{D}_k \subset D_{k+1}$) up to D . We then reduce the problem to prove that for any convex domain D , $P_x\{B_{jt/n} \in D; j = 1, 2, \dots, n\}$ is log-concave on D as a function of x , for all $t > 0$ and all n . This, however, is a multiple convolution of Gaussians with the indicator function of the set D . Since the Gaussian $p_t^2(x)$ is log-concave for all $t > 0$ and the indicator function of a convex domain is log-concave, the result follows from the fact that convolutions of log-concave functions are log-concave. Using right continuity of paths, we can try to repeat this argument for α -stable processes. However, this time the argument breaks down right at the end. For example, if $\alpha = 1$ the density for the Cauchy process, $p_t^1(x, y) = p_t^1(x - y)$, is not log-concave for all t . The obvious variation of this argument using the fact that $X_t^\alpha = B_{2\sigma_t}$, where σ_t is a stable subordinator of index $\alpha/2$ independent of B_t , also fails basically due to the fact that the sum of log-concave functions is not necessarily log-concave.

There is however, a substitute for log-concavity which gives some insight into the shape of the ground state eigenfunction. We call this property “mid-concavity”.

Definition 1.1. *Let $D \subset \mathbb{R}^d$ be a convex domain which is symmetric relative to each coordinate axes. Let J be a line segment in D parallel to the x_1 -axis which intersects the boundary ∂D only at the two points $(-a_1, a_2, \dots, a_d)$, (a_1, a_2, \dots, a_d) , $a_1 > 0$. We will say that the function $F : D \rightarrow \mathbb{R}$, is mid-concave on J if it is concave on the segment (half of J) from the point $(-a_1/2, a_2, \dots, a_d)$ to $(a_1/2, a_2, \dots, a_d)$. The function is mid-concave along the x_1 -axis if it is mid-concave on every such segment contained in D which is parallel to the x_1 -axis. A similar definition applies for mid-concavity along the x_2 -axis, \dots , x_d -axis. The function is mid-concave on D if it is mid-concave along each coordinate axes.*

Our main result in this paper is the following

Theorem 1.1. *Let $Q = (-a_1, a_1) \times (-a_2, a_2) \times \dots \times (-a_d, a_d)$, $0 < a_i < \infty$ for all $i = 1, 2, \dots, d$, be a rectangle in \mathbb{R}^d . The ground state eigenfunction φ_1^α for the symmetric stable process of index $0 < \alpha < 2$*

is mid-concave on Q . In addition, if $x = (x_1, \dots, x_n) \in Q$, then

$$(1.4) \quad \frac{\partial}{\partial x_i} \varphi_1^\alpha(x) \geq 0, \text{ if } x_i < 0, \text{ and } \frac{\partial}{\partial x_i} \varphi_1^\alpha(x) \leq 0, \text{ if } x_i > 0.$$

Using arguments of multiple integrals as described above, we will show that Theorem 1.1 follows from

Theorem 1.2. *Let Q be a rectangle in \mathbb{R}^d . Let $0 < t_1 < t_2 < \dots < t_n < \infty$. The function*

$$(1.5) \quad F(x) = P_x\{X_{t_1}^\alpha \in Q, \dots, X_{t_n}^\alpha \in Q\}$$

is mid-concave in Q for any $0 < \alpha \leq 2$. In addition, if $x = (x_1, \dots, x_n) \in Q$, then

$$(1.6) \quad \frac{\partial}{\partial x_i} F(x) \geq 0, \text{ if } x_i < 0, \text{ and } \frac{\partial}{\partial x_i} F(x) \leq 0, \text{ if } x_i > 0.$$

Remark 1.1. *It is important to note here that Theorem 1.2 is new even in the Brownian motion case ($\alpha = 2$). Indeed, as we shall see, the case $\alpha = 2$ implies the general case by subordination.*

If we consider the eigenfunction for the Laplacian in the unit disk \mathbb{D} in the plane, one can show, by analysis of the Bessel function, that such a function is not concave in \mathbb{D} but it is *mid-concave*. Also, it may be tempting to conjecture that for any symmetric domain in the plane the eigenfunction is *mid-concave*. This, however, is not the case, even for the Brownian motion, as we will show at the end of the paper.

The paper is organized as follows. In §2, we prove that the multiple convolutions of Gaussians in the interval $(-1, 1)$ is *mid-concave*. In §3, we show how this and subordination implies Theorem 1.2. Here we also show that full concavity fails for general multiple integrals and that *mid-concavity* fails in general symmetric domains in the plane.

2. MID-CONCAVITY FOR BROWNIAN MOTION

Let

$$p_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$$

be the Gaussian density in one dimension. With the notation of the introduction, we have $p_t^2(x, y) = p_{2t}(x - y)$.

Proposition 2.1. *Let $n = 1, 2, \dots$ and let t_1, t_2, \dots, t_n be real numbers in $(0, \infty)$. For $x \in (-1, 1)$ define*

$$(2.1) \quad \Phi_n(x) = \int_{-1}^1 \dots \int_{-1}^1 \prod_{i=1}^n p_{t_i}(x_{i-1} - x_i) dx_1 \dots dx_n,$$

where $x_0 = x$. The function $\Phi_n(x)$ is mid-concave on $(-1, 1)$. That is, $\Phi_n(x)$ is concave on $(-\frac{1}{2}, \frac{1}{2})$.

Clearly $\Phi_n(x)$ is a positive even function on $[-1, 1]$. Integrating by parts we obtain

$$\begin{aligned}
 -\frac{\partial}{\partial x} \Phi_n(x) &= \frac{1}{\sqrt{2\pi t_n}} \int_{-1}^1 \left(\frac{\partial}{\partial y} e^{-\frac{(y-x)^2}{2t_n}} \right) \Phi_{n-1}(y) dy \\
 (2.2) \quad &= \frac{\Phi_{n-1}(1)}{\sqrt{2\pi t_n}} \left(e^{-\frac{(1-x)^2}{2t_n}} - e^{-\frac{(1+x)^2}{2t_n}} \right) \\
 &\quad - \frac{1}{\sqrt{2\pi t_n}} \int_{-1}^1 e^{-\frac{(y-x)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy.
 \end{aligned}$$

Notice that for all $t > 0$,

$$(2.3) \quad \left(e^{-\frac{(1-x)^2}{2t}} - e^{-\frac{(1+x)^2}{2t}} \right) = e^{\frac{-(1-x)^2}{2}} \left(1 - e^{-\frac{2x}{t}} \right),$$

is a positive increasing function on $[0, 1]$.

Lemma 2.1. *The function $\Phi_n(x)$ is decreasing on $(0, 1)$ for all $n \geq 1$.*

Proof. We argue by induction. If $n = 1$, then

$$\begin{aligned}
 \frac{\partial}{\partial x} \Phi_1(x) &= \frac{1}{\sqrt{2\pi t_1}} \int_{-1}^1 \frac{-\partial}{\partial y} e^{-\frac{(y-x)^2}{2t_1}} dy \\
 (2.4) \quad &= \frac{1}{\sqrt{2\pi t_1}} \left(-e^{-\frac{(1-x)^2}{2t_1}} + e^{-\frac{(1+x)^2}{2t_1}} \right) \\
 &< 0,
 \end{aligned}$$

for all $x \in (0, 1)$. Thus $\Phi_1(x)$ is decreasing on $(0, 1)$.

Let us assume that $\Phi_{n-1}(x)$ is decreasing on $(0, 1)$. That is, suppose that

$$\frac{\partial}{\partial x} \Phi_{n-1}(x) \leq 0,$$

for all $x \in (0, 1)$. Because of (2.3), it is enough to prove that

$$(2.5) \quad \frac{-1}{\sqrt{2\pi t_n}} \int_{-1}^1 e^{-\frac{(y-x)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy \geq 0.$$

By symmetry

$$\frac{\partial}{\partial y} \Phi_{n-1}(y) = -\frac{\partial}{\partial y} \Phi_{n-1}(-y).$$

On the other hand, if $x > 0$ then

$$e^{-\frac{(x-y)^2}{2t}} \geq e^{-\frac{(x+y)^2}{2t}},$$

for all $t, y > 0$. Hence for all $y > 0$,

$$\begin{aligned} -e^{-\frac{(x-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) - e^{-\frac{(x+y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(-y) = \\ -e^{-\frac{(x-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) + e^{-\frac{(x+y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) = \\ \left(-e^{-\frac{(x-y)^2}{2t_n}} + e^{-\frac{(x+y)^2}{2t_n}} \right) \frac{\partial}{\partial y} \Phi_{n-1}(y) \geq 0. \end{aligned}$$

Integrating this inequality on $[0, 1]$ we obtain (2.5). \square

Notice that

$$\begin{aligned} \frac{\partial^2}{\partial x^2} \Phi_1(x) &= \frac{1}{\sqrt{2\pi t_1}} \int_{-1}^1 \frac{\partial^2}{\partial y^2} e^{-\frac{(y-x)^2}{2t_1}} dy \\ (2.6) \quad &= \frac{1}{t_1 \sqrt{2\pi t_1}} \left(-(1-x)e^{-\frac{(1-x)^2}{2t_1}} - (1+x)e^{-\frac{(1+x)^2}{2t_1}} \right) \\ &< 0, \end{aligned}$$

for all $x \in (-1, 1)$. Thus $\Phi_1(x)$ is concave in $(-1, 1)$. We will know prove that $\Phi_n(x)$ is concave in $(-\frac{1}{2}, \frac{1}{2})$.

Lemma 2.2. *If $0 \leq x \leq \frac{1}{2}$, then for all $n \geq 1$,*

$$\frac{\partial}{\partial x} \Phi_n(x) \geq \frac{\partial}{\partial x} \Phi_n(1-x).$$

Proof. By (2.6) the result is true for $n = 1$. Let us assume that the result is true for $n - 1$. Let

$$\psi_n(x) = \frac{-1}{\sqrt{2\pi t_n}} \int_{-1}^1 e^{-\frac{(y-x)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy.$$

Because of (2.3), it is enough to prove that

$$(2.7) \quad \psi_n(1-x) \geq \psi_n(x).$$

Let $y \in (-1, 0)$, then

$$1 - x - y \geq x - y \geq 0.$$

Thus

$$e^{-\frac{(1-x-y)^2}{2t_n}} \leq e^{-\frac{(x-y)^2}{2t_n}}.$$

Since

$$-\frac{\partial}{\partial y} \Phi_{n-1}(y) \leq 0,$$

for all $y < 0$, it follows that

$$-\int_{-1}^0 e^{-\frac{(x-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy \leq -\int_{-1}^0 e^{-\frac{(1-x-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy.$$

To simplify notation let

$$\phi(y) = -\frac{\partial}{\partial y} \Phi_{n-1}(y).$$

Let $y \in (0, \frac{1}{2})$, and consider $\hat{y} = 1 - y$. Notice that $\hat{y} \in (\frac{1}{2}, 1)$ and

$$\frac{1}{2} - y = \hat{y} - \frac{1}{2}.$$

By induction,

$$0 \leq \phi(y) \leq \phi(\hat{y}).$$

On the other hand,

$$\begin{aligned} e^{-\frac{(x-y)^2}{2t_n}} &= e^{-\frac{(\hat{x}-\hat{y})^2}{2t_n}}, \\ e^{-\frac{(x-\hat{y})^2}{2t_n}} &= e^{-\frac{(\hat{x}-y)^2}{2t_n}}, \\ e^{-\frac{(x-y)^2}{2t_n}} &\geq e^{-\frac{(x-\hat{y})^2}{2t_n}}. \end{aligned}$$

Thus

$$\left(e^{-\frac{(x-y)^2}{2t_n}} - e^{-\frac{(\hat{x}-y)^2}{2t_n}} \right) \phi(y) \leq \left(e^{-\frac{(\hat{x}-\hat{y})^2}{2t_n}} - e^{-\frac{(x-\hat{y})^2}{2t_n}} \right) \phi(\hat{y}),$$

and we conclude that

$$e^{-\frac{(x-y)^2}{2t_n}} \phi(y) + e^{-\frac{(x-\hat{y})^2}{2t_n}} \phi(\hat{y}) \leq e^{-\frac{(\hat{x}-y)^2}{2t_n}} \phi(y) + e^{-\frac{(\hat{x}-\hat{y})^2}{2t_n}} \phi(\hat{y}).$$

Integrating over $(0, \frac{1}{2})$ we obtained that

$$-\int_0^1 e^{-\frac{(x-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy \leq -\int_0^1 e^{-\frac{(1-x-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy,$$

the desired result immediately follows. \square

Lemma 2.3. *If $0 \leq x < u \leq \frac{1}{2}$, then for all $n \geq 1$,*

$$\frac{\partial}{\partial x} \Phi_n(x) \geq \frac{\partial}{\partial x} \Phi_n(u).$$

Proof. By (2.6) the result is true for $n = 1$. Let us assume that the result is true for $n - 1$. As in Lemma 2.1, we let

$$\psi_n(x) = \frac{-1}{\sqrt{2\pi t_n}} \int_{-1}^1 e^{-\frac{(y-x)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy$$

and

$$\phi(y) = -\frac{\partial}{\partial y} \Phi_{n-1}(y).$$

By (2.3), it is enough to prove that

$$(2.8) \quad \psi_n(u) \geq \psi_n(x).$$

Let $y \in (-1, 0)$, then $|u - y| \geq |x - y|$. Thus

$$e^{-\frac{(u-y)^2}{2t_n}} \leq e^{-\frac{(x-y)^2}{2t_n}}.$$

Lemma 2.1 implies that

$$-\int_{-1}^0 e^{-\frac{(x-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy \leq -\int_{-1}^0 e^{-\frac{(u-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy.$$

Let $m = \frac{x+u}{2}$. For all $y \in (0, m)$, define $\tilde{y} = x + u - y$. Notice that $\tilde{y} \in (m, x+u)$ and that

$$|x - y| = |u - \tilde{y}|.$$

We can easily check that

$$\begin{aligned} e^{-\frac{(x-y)^2}{2t_n}} &= e^{-\frac{(u-\tilde{y})^2}{2t_n}}, \\ e^{-\frac{(x-\tilde{y})^2}{2t_n}} &= e^{-\frac{(u-y)^2}{2t_n}}, \\ e^{-\frac{(x-y)^2}{2t_n}} &\geq e^{-\frac{(x-\tilde{y})^2}{2t_n}}, \end{aligned}$$

for all $y \in (0, m)$. We claim that

$$(2.9) \quad \phi(y) \leq \phi(\tilde{y}).$$

This follows immediately from the induction hypothesis if $\tilde{y} \leq \frac{1}{2}$. On the other hand, if $\tilde{y} = (x+u) - y \geq \frac{1}{2}$, then

$$1 - (x+u) + y \leq \frac{1}{2}, \text{ and } y \leq 1 - (x+u) + \tilde{y}.$$

Lemma 2.2 and the induction hypothesis imply that

$$0 \leq \phi(y) \leq \phi(1 - (x+u) + y) \leq \phi((x+u) - y) = \phi(\tilde{y}).$$

Thus

$$e^{-\frac{(x-y)^2}{2t_n}} \phi(y) + e^{-\frac{(x-\tilde{y})^2}{2t_n}} \phi(\tilde{y}) \leq e^{-\frac{(u-y)^2}{2t_n}} \phi(y) + e^{-\frac{(u-\tilde{y})^2}{2t_n}} \phi(\tilde{y}).$$

Integrating over $(0, m)$ we obtained that

$$-\int_0^{x+u} e^{-\frac{(x-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy \leq -\int_0^{x+u} e^{-\frac{(u-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy.$$

Finally if $y \in [x+u, 1]$ then

$$e^{-\frac{(x-y)^2}{2t_n}} \leq e^{-\frac{(u-y)^2}{2t_n}}.$$

Therefore

$$-\int_{x+u}^1 e^{-\frac{(x-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy \leq -\int_{x+u}^1 e^{-\frac{(u-y)^2}{2t_n}} \frac{\partial}{\partial y} \Phi_{n-1}(y) dy.$$

□

By symmetry, Proposition 2.1 follows from Lemma 2.3. The following is an immediate corollary of Proposition 2.1.

Corollary 2.1. *Let B_t be one dimensional Brownian motion and set $I = (-a, a)$, For $0 < t_1 < t_2 < \dots < t_n$, the function*

$$(2.10) \quad F(x) = P_x \{B_{t_1} \in I, B_{t_2} \in I, \dots, B_{t_n} \in I\}$$

is mid-concave in I . In addition, if $x \in I$, then

$$(2.11) \quad F'(x) \geq 0, \text{ if } x < 0, \text{ and } F'(x) \leq 0, \text{ if } x > 0.$$

Proof. By the Markov property,

$$(2.12) \quad F(x) = \int_{-a}^a \dots \int_{-a}^a \prod_{i=1}^n p_{t_i-t_{i-1}}(x_{i-1} - x_i) dx_1 \dots dx_n,$$

where $x_0 = x$ and $t_0 = 0$. This is exactly the same expression as in Lemma 2.1 and Proposition 2.1 except for the fact that the interval $(-1, 1)$ has been replaced by the interval $(-a, a)$. The proof of the proposition is the same for this case and the corollary follows. □

Corollary 2.2. *Let B_t be Brownian motion in \mathbb{R}^d and let $Q = I_1 \times I_2 \times \dots \times I_d$ where $I_i = (-a_i, a_i)$, be a rectangle in \mathbb{R}^d . For $0 < t_1 < t_2 < \dots < t_n$, the function*

$$(2.13) \quad F(x) = P_x \{B_{t_1} \in Q, B_{t_2} \in Q, \dots, B_{t_n} \in Q\}$$

is mid-concave in Q . In addition, if $x = (x_1, x_2, \dots, x_d) \in Q$, then

$$(2.14) \quad \frac{\partial}{\partial x_i} F(x) \geq 0, \text{ if } x_i < 0, \text{ and } \frac{\partial}{\partial x_i} F(x) \leq 0, \text{ if } x > 0.$$

Proof. With $x = (x_1, x_2, \dots, x_d)$ and $B_t = (B_t^1, B_t^2, \dots, B_t^d)$, it follows by independence that

$$F(x) = \prod_{i=1}^d P_{x_i} \{ B_{t_1}^i \in I_i, B_{t_2}^i \in I_i, \dots, B_{t_n}^i \in I_i \}$$

and the conclusion of the corollary follows from Corollary 2.1 and our definition of *mid-concavity* for domains in \mathbb{R}^d . \square

3. MID-CONCAVITY FOR STABLE PROCESSES

In this section we prove Theorems 1.1 and 1.2. First, let us recall that for $0 < \alpha < 2$ the symmetric stable process X_t^α in \mathbb{R}^d has the representation

$$(3.1) \quad X_t^\alpha = B_{2\sigma_t},$$

where σ_t is a stable subordinator of index $\alpha/2$ independent of B_t (see [6]). Thus

$$(3.2) \quad p_t^\alpha(x - y) = \int_0^\infty p_s^2(x - y) g_{\alpha/2}(t, s) ds,$$

where $g_{\alpha/2}(t, s)$ is the transition density of σ_t and

$$p_{t/2}^2(x - y) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{|x-y|^2}{2t}}.$$

Now, let Q and t_1, t_2, \dots, t_n be as in the statement of Theorem 1.2. Set $x_0 = x$ and $t_0 = 0$. Using the Markov property of the stable process X_t^α , the subordination formula (3.2), Fubini's theorem, and the Markov property of the Brownian motion, in this order, we obtain,

$$\begin{aligned} F(x) &= P_x \{ X_{t_1}^\alpha \in Q, \dots, X_{t_n}^\alpha \in Q \} \\ &= \int_Q \dots \int_Q \prod_{i=1}^n p_{t_i - t_{i-1}}^\alpha(x_{i-1} - x_i) dx_1 \dots dx_n \\ &= \int_0^\infty \dots \int_0^\infty \left(\int_Q \dots \int_Q \prod_{i=1}^n p_{s_i}^2(x_{i-1} - x_i) dx_1 \dots dx_n \right) \\ &\quad \times \prod_{i=1}^n g_{\alpha/2}(t_i - t_{i-1}, s_i) ds_1 \dots ds_n \\ &= \int_0^\infty \dots \int_0^\infty P_x \{ B_{2s_1} \in Q, B_{2(s_1+s_2)} \in Q, \dots, B_{2(s_1+s_2+\dots+s_n)} \in Q \} \\ &\quad \times \prod_{i=1}^n g_{\alpha/2}(t_i - t_{i-1}, s_i) ds_1 \dots ds_n. \end{aligned}$$

Since the function

$$P_x\{B_{2s_1} \in Q, B_{2(s_1+s_2)} \in Q, \dots, B_{2(s_1+s_2+\dots+s_n)} \in Q\}$$

is *mid-concave* and satisfies the monotonicity property (2.14), by Corollary 2.2, so is the integral against the densities $g_{\alpha/2}(t_i - t_{i-1}, s_i)$ and this completes the proof of Theorem 1.2.

With Theorem 1.2 proved, we argue as in the proof of the *log-concavity* for Brownian motion discussed in the introduction. Recall that φ_1^α is the ground state eigenfunction for the stable process of index α , $\alpha \in (0, 2)$, killed upon leaving Q and λ_1^α is its eigenvalue. Let τ_Q^α be the first exit time of the symmetric stable process from Q . Since Q is certainly intrinsically ultracontractive, see [17], we have that

$$(3.3) \quad \varphi_1^\alpha(x) = \lim_{t \rightarrow \infty} e^{\lambda_1^\alpha t} P_x\{\tau_Q^\alpha > t\}.$$

The convergence is uniform for $x \in Q$. Thus to prove *mid-concavity* for $\varphi_1^\alpha(x)$ it is enough to prove mid-concavity for $P_x\{\tau_Q^\alpha > t\}$. By the right continuity of the sample paths, we have,

$$(3.4) \quad \begin{aligned} P_x\{\tau_Q^\alpha > t\} &= P_z\{X_s^\alpha \in Q, 0 \leq s \leq t\} \\ &= \lim_{n \rightarrow \infty} P_x\{X_{\frac{it}{n}}^\alpha \in Q, i = 1, \dots, n\}. \end{aligned}$$

Theorem 1.1 now follows from this and Theorem 1.2.

We remark that in the case of Brownian motion, there is an extra approximation by an increasing sequence of domains in passing from the first equality to the second in (3.4). This is not needed for our stable processes since, as explained in [8], Lemma 6, for any domain $D \subset \mathbb{R}^d$ with Lipschitz boundary,

$$P_x\{X_{\tau_D}^\alpha \in \partial D\} = 0 \text{ for } x \in D.$$

The above argument applies not only to symmetric stable processes but also to any other process which is obtained by subordination of Brownian motion. In particular, the above results hold for the so called “relativistic” Brownian motion and “relativistic” α -stable processes studied in [19].

It is of course natural to ask if the function of Proposition 2.1 is concave in the whole interval $(-1, 1)$ for all n and all t_i . Notice that, thanks to the proof of Lemma 2.2, this is the case for $n = 1$. If this were the case, it would show that the same is true for the function $P_x\{\tau_Q^\alpha > t\}$ and hence for the function φ_1^α , as desired. Unfortunately, this is not the case.

Proposition 3.1. *Let*

$$(3.5) \quad \Phi_n(x) = \int_{-1}^1 \cdots \int_{-1}^1 \prod_{i=1}^n p_{t_i}(x_{i-1} - x_i) dx_1 \dots dx_n,$$

where $x_0 = x$. Then there exist a positive integer n and real numbers t_1, t_2, \dots, t_n in $(0, \infty)$ such that the function $\Phi_n(x)$ is not concave on $(-1, 1)$.

Proof. We may replace, to simplify certain notation below, the interval $(-1, 1)$ by the interval $(0, \pi)$. Fix t and s both positive. Let $t_1 = t$ and $t_2 = \dots = t_n = \frac{s}{n-1}$. If the function $\Phi_n(x)$ is concave on $(0, \pi)$ for all n with these chosen t_1, t_2, \dots, t_n , letting $n \rightarrow \infty$ we see that the function

$$(3.6) \quad \int_0^\pi p_t(x - y) P_y \{ \tau_{(0,\pi)} > s \} dy$$

is also concave on $(0, \pi)$. Here we have used $\tau_{(0,\pi)}$ to denote the first exit time of Brownian motion from the interval. We have

$$(3.7) \quad \lim_{s \rightarrow \infty} e^{\lambda_1 s} P_y \{ \tau_{(0,\pi)} > s \} = c \sin(y),$$

uniformly for $y \in (0, \pi)$, where $c > 0$ and $\lambda_1 = 1$ (the first eigenvalue for $(0, \pi)$). It follows that for each $t > 0$, the function

$$(3.8) \quad F_t(x) = \int_0^\pi p_t(x - y) \sin(y) dy$$

must also be concave on $(0, \pi)$.

We will now show that the function $F_t(x)$ is not concave. Without any difficulty we may differentiate under the integral to obtain that

$$(3.9) \quad F_t''(x) = \frac{1}{\sqrt{2\pi} t^{5/2}} \int_0^\pi [(x - y)^2 - t] e^{\frac{-(x-y)^2}{2t}} \sin(y) dy.$$

Taking $x = 0$ and using the elementary inequality

$$y - \frac{y^3}{3!} \leq \sin(y) \leq y$$

valid for all $y > 0$, we see that $F_t''(0)$ is equal to

$$\begin{aligned}
& \frac{1}{\sqrt{2\pi t^{5/2}}} \int_0^\pi (y^2 - t) e^{-\frac{y^2}{2t}} \sin(y) dy \\
&= \frac{1}{\sqrt{2\pi t^{5/2}}} \left(\int_0^\pi y^2 e^{-\frac{y^2}{2t}} \sin(y) dy - t \int_0^\pi e^{-\frac{y^2}{2t}} \sin(y) dy \right) \\
&\geq \frac{1}{\sqrt{2\pi t^{5/2}}} \left(\int_0^\pi y^3 e^{-\frac{y^2}{2t}} dy - \frac{1}{3!} \int_0^\pi y^5 e^{-\frac{y^2}{2t}} dy - t \int_0^\pi y e^{-\frac{y^2}{2t}} dy \right) \\
&= \frac{1}{\sqrt{2\pi t^{5/2}}} \left(t^2 \int_0^{\frac{\pi}{\sqrt{t}}} y^3 e^{-\frac{y^2}{2}} dy - \frac{t^3}{3!} \int_0^{\frac{\pi}{\sqrt{t}}} y^5 e^{-\frac{y^2}{2}} dy - t^2 \int_0^{\frac{\pi}{\sqrt{t}}} y e^{-\frac{y^2}{2}} dy \right) \\
&= \frac{1}{\sqrt{2\pi t}} \left(\int_0^{\frac{\pi}{\sqrt{t}}} y^3 e^{-\frac{y^2}{2}} dy - \frac{t}{3!} \int_0^{\frac{\pi}{\sqrt{t}}} y^5 e^{-\frac{y^2}{2}} dy - \int_0^{\frac{\pi}{\sqrt{t}}} y e^{-\frac{y^2}{2}} dy \right).
\end{aligned}$$

Since

$$\frac{t}{3!} \int_0^{\frac{\pi}{\sqrt{t}}} y^5 e^{-\frac{y^2}{2}} dy \rightarrow 0,$$

$$\int_0^{\frac{\pi}{\sqrt{t}}} y^3 e^{-\frac{y^2}{2}} dy \rightarrow 2,$$

and

$$\int_0^{\frac{\pi}{\sqrt{t}}} y e^{-\frac{y^2}{2}} dy \rightarrow 1,$$

as $t \rightarrow 0^+$, we see that $F_t''(0)$ is positive for sufficiently small t . By continuity, we have that $F_t''(x) > 0$ for sufficiently small $x \in (0, \pi)$ and sufficiently small t . This, of course, contradicts the concavity of the function and shows that $\Phi_n(x)$ is not concave.

□

Of course, it may still be the case that the function $\Phi_n(x)$ is concave on the whole interval when we restrict to a sequence of times satisfying $t_1 = t_2 = \dots = t_n$ and substitute $p_{t_i}(x)$ by $p_{t_i}^\alpha(x)$, which is what is needed for our applications (Conjecture 1.1). That is, the following conjecture may still be true.

Conjecture 3.1. *Let $I = (-1, 1)$ and let n be a positive integer. If $t_i = \frac{it}{n}$ for $1 \leq i \leq n$, then the function*

$$(3.10) \quad F(x) = P_x \{ X_{t_1}^\alpha \in I, \dots, X_{t_n}^\alpha \in I \}$$

is concave on I .

A natural question is whether φ_1^α is *mid-concave* for any symmetric, convex domain in the plane. We will now show that for a large enough

rhombus and $\alpha = 2$ (Brownian motion), this is not the case. Below we use $\lambda_1(D)$ and φ_D to denote the first eigenvalue for the domain D and its corresponding eigenfunction, respectively, for the Brownian motion. We also denote the first exit time of the Brownian motion from a domain D by τ_D .

Proposition 3.2. *For $n \geq 1$, set*

$$D(n) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \in (-n, n), x_2 \in \left(-1 + \frac{|x_1|}{n}, 1 - \frac{|x_1|}{n}\right) \right\}.$$

There exists an n large enough such that $\varphi_{D(n)}$ is not mid-concave on $D(n)$.

Proof. The rectangle

$$R(n) = (-\sqrt{n}, \sqrt{n}) \times \left(-1 + \frac{1}{\sqrt{n}}, 1 - \frac{1}{\sqrt{n}}\right)$$

is a subset of $D(n)$ and therefore,

$$\lambda_1(D(n)) < \lambda_1(R(n)) = \frac{\pi^2}{(2 - 2/\sqrt{n})^2} + \frac{\pi^2}{(2\sqrt{n})^2} \leq \frac{\pi^2}{4} \left(1 + \frac{3}{\sqrt{n}}\right),$$

for n large enough. Now, for any $a \in (0, 1/2)$, consider the subset of $D(n)$ define by

$$Q(a, n) = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \in (an, n), x_2 \in \left(-1 + \frac{|x_1|}{n}, 1 - \frac{|x_1|}{n}\right) \right\}.$$

Since

$$\Delta \varphi_{D(n)} + \lambda_1(D(n)) \varphi_{D(n)} = 0$$

in $Q(a, n)$ and $\lambda_1(D(n)) < \lambda_1(Q(a, n))$. That is, $\varphi_{D(n)}$ is a q -harmonic function with $q = \lambda_1(D(n))$. The Feynman–Kac formula gives that for any $x \in Q(a, n)$,

$$\begin{aligned} \varphi_{D(n)}(x) &= E_x \left[e^{\lambda_1(D(n)) \tau_{Q(a,n)}} \varphi_{D(n)}(B(\tau_{Q(a,n)})) \right] \\ (3.11) \quad &\leq \varphi_{D(n)}(0) E_x \left[e^{\lambda_1(D(n)) \tau_{Q(a,n)}}; B(\tau_{Q(a,n)}) \in D(n) \setminus Q(a, n) \right]. \end{aligned}$$

Of course,

$$\varphi_{D(n)}(0) = \max \{ \varphi_{D(n)}(x) : x \in D \},$$

by symmetry. Let $p(a) = (1 - a/2)^2/(1 - a)^2$ and $q(a)$ be such that $1/p(a) + 1/q(a) = 1$. Note that $p(a) > 1$ so $q(a) > 0$. By Hölder's inequality the expression in (3.11) is bounded above by

$$\begin{aligned} \varphi_{D(n)}(0) &\left(E_x \left[e^{\lambda_1(D(n)) \tau_{Q(a,n)} p(a)} \right] \right)^{1/p(a)} \\ &\times \left(P_x \left[B(\tau_{Q(a,n)}) \in D(n) \setminus Q(a, n) \right] \right)^{1/q(a)}. \end{aligned}$$

Since

$$Q(a, n) \subset (-\infty, \infty) \times (-1 + a, 1 - a)$$

we have that for any $x \in Q(a, n)$,

$$\begin{aligned} & E_x \left[e^{\lambda_1(D(n)) \tau_{Q(a,n)} p(a)} \right] \\ & \leq E_0 \left[\exp \left((\pi^2/4) (1 + 3/\sqrt{n}) p(a) \tau_{(-1+a, 1-a)} \right) \right] \\ & = E_0 \left[\exp \left((\pi^2/4) (1 + 3/\sqrt{n}) p(a) (1-a)^2 \tau_{(-1, 1)} \right) \right] \\ & = E_0 \left[\exp \left((\pi^2/4) (1 + 3/\sqrt{n}) (1-a/2)^2 \tau_{(-1, 1)} \right) \right]. \end{aligned}$$

By a simple calculation we see that $(1 + 3/\sqrt{n})(1 - a/2) \leq 1$ when $n \geq (6 - 3a)^2/a^2$. For such n , we have

$$\begin{aligned} & \left(E_x \left[e^{\lambda_1(D(n)) \tau_{Q(a,n)} p(a)} \right] \right)^{1/p(a)} \\ & \leq \left(E_0 \left[\exp \left((\pi^2/4) (1 - a/2) \tau_{(-1, 1)} \right) \right] \right)^{1/p(a)} = C_1(a). \end{aligned}$$

Using the fact that $\frac{\pi^2}{4}$ is the eigenvalue for the interval $(-1, 1)$, we have that for any $c \in (0, 1)$, $E_0[\exp(c \tau_{(-1, 1)} \pi^2/4)] < \infty$. Thus for any $a \in (0, 1/2)$ we have $C_1(a) < \infty$.

By standard results for Brownian motion (or the trivial estimate of the harmonic measure in the strip obtained by conformal mapping to the disk), for any $b \geq 0$ and $x_1 > b$ we have

$$P_{(x_1, 0)} \left[B(\tau_{(b, \infty) \times (-1, 1)}) \in (-\infty, b) \times (-1, 1) \right] \leq C_2 e^{-\frac{\pi}{2}(x_1 - b)},$$

where $C_2 > 0$ is an absolute constant.

Note that $x = (2an, 0) \in Q(a, n)$. It follows that

$$P_{(2an, 0)} \left[B(\tau_{Q(a,n)}) \in D(n) \setminus Q(a, n) \right] \leq C_2 e^{-\frac{\pi}{2}an}.$$

Now choose $a = 1/8$. For such a we have $(2an, 0) = (n/4, 0)$. For $n \geq (6 - 3a)^2/a^2$ we have

$$(3.12) \quad \varphi_{D(n)}(n/4, 0) \leq \varphi_{D(n)}(0, 0) C_1(1/8) \left[C_2 e^{-\frac{\pi}{16}n} \right]^{\frac{1}{q(1/8)}}.$$

If $\varphi_{D(n)}$ were *mid-concave*, we would have

$$\varphi_{D(n)}(n/4, 0) \geq \frac{1}{2} [\varphi_{D(n)}(0, 0) + \varphi_{D(n)}(n/2, 0)] \geq \frac{1}{2} \varphi_{D(n)}(0, 0).$$

However, by (3.12) for large enough n we have that $\varphi_{D(n)}(n/4, 0)$ is smaller than $\varphi_{D(n)}(0, 0)/2$. Thus $\varphi_{D(n)}$ is not *mid-concave*. Indeed, the same argument shows that for any $c \in (0, 1)$ there exists an n large enough such that $\varphi_{D(n)}$ is not concave on the interval with endpoints $(-cn, 0), (cn, 0)$. \square

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MATHEMATICS DEPARTMENT, PURDUE UNIVERSITY, WEST LAFAYETTE, IN 47907

E-mail address: banuelos@math.purdue.edu

INSTITUTE OF MATHEMATICS, WROCŁAW UNIVERSITY OF TECHNOLOGY, WYB. WYSPIANSKIEGO 27, 50-370 WROCŁAW, POLAND

E-mail address: tkulczyc@im.pwr.wroc.pl

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF UTAH, 155 S. 1400 E. SALT LAKE CITY, UT, 84112-0090

E-mail address: mendez@math.utah.edu